Matrices

1. Introduction of Matrix

- The table shows the expenses of Adam and Aarif for a month.

Expenses	Adam	Aarif
Bus fare	10	13
Stationary	5	4
Meals	9	12

- The information can be represent as a rectangular array of numbers as below:
$\begin{pmatrix} 10 & 13 \\ 5 & 4 \\ 9 & 12 \end{pmatrix}$

- This rectangular array of numbers in rows and columns is called a**matrix**(plural: matrices).

- This matrix has 3 rows and 2 columns, we say that it has **order**, or **dimension**, 3×2 (read as 3 by 2), where the number of rows is specified first.

- The numbers in a matrix is called its **elements** or **entries**.

- A matrix that has only one column is called a **column matrix**. Example: $\begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$

- A matrix that has only one row is called a **row matrix**. Example: $\begin{pmatrix} 2 & 5 \end{pmatrix}, \begin{pmatrix} 2 & -3 & 5 \end{pmatrix}$

- If the number rows and columns are equal, the matrix is called **square matrix**. Example: $\begin{pmatrix} -2 & 3 \\ 1 & 10 \end{pmatrix}$, $\begin{pmatrix} 1 & -2 & 4 \\ 2 & 5 & 6 \\ 3 & 7 & -5 \end{pmatrix}$

- Two matrices are equal (**equal matrix**) if they have the same order and of their corresponding elements are equal. Given $A = \begin{pmatrix} 1 & 4 \\ -2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 0 \\ 1 & -2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 4 \\ -2 & 0 \end{pmatrix}$, so $A = C$, but $A \neq B$

- A matrix with all the elements are zeros is call **zero** or **null matrix**, and usually denoted by **O**.
$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- **Transpose matrix**, denoted by A' ,A^t or A^T is matrix when the row and column's elements are exchange. For example,

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 5 & -2 & 4 \end{pmatrix} \quad A' = \begin{pmatrix} 1 & 5 \\ 3 & -2 \\ 2 & 4 \end{pmatrix}$$

Example 1: If $\begin{pmatrix} a+1 & a+c \\ c & d+2 \end{pmatrix} = \begin{pmatrix} -2 & -b \\ b-d & 1 \end{pmatrix}$, find the value of a, b, c and d .

2. Addition and Subtraction of Matrix

- For two matrices A and B of the same order,
  $\Rightarrow$ the matrix $A+B$ is obtained by adding the corresponding elements of A and B.
  $\Rightarrow$ the matrix $A-B$ is obtained by subtracting the corresponding elements of B from A.

Example 2: Calculate the following

(a) $\begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 1 & -6 \\ 1 & 3 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 5 \\ 3 & 4 \\ 1 & -5 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 3 & 4 \\ 2 & -3 \end{pmatrix}$

3. Multiplication of Matrices

- Scalar multiplication: a matrix and areal number k (scalar), the matrix kA is obtained by multiplying each elements in A by k.

Example 3: Calculate the following

(a) $2\begin{pmatrix} 1 & -1 \\ 4 & 3 \end{pmatrix} - 3\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & 2 & 0 \\ 4 & 5 & 1 \end{pmatrix} + 4\begin{pmatrix} 2 & 0 & 1 \\ 3 & -1 & 4 \end{pmatrix}$

Example 4: Given that $A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} -1 & 2 & 3 \\ 5 & 1 & -1 \end{pmatrix}$. *If* $3X - 2B = X + 4A$, *find matrix X.*

Example 5. Find the matrix of $2A - 3B$ *if* $A = \begin{pmatrix} 2 & 0 \\ -3 & 5 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -1 \\ 2 & -4 \end{pmatrix}$.

Exercise 1

1. $\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

2. $\begin{pmatrix} -2 & 3 \\ 1 & 10 \end{pmatrix} + \begin{pmatrix} 5 & -3 \\ 0 & -7 \end{pmatrix}$

3. $\begin{pmatrix} -1 & 13 \\ 10 & 0 \end{pmatrix} - \begin{pmatrix} -5 & -10 \\ 5 & -7 \end{pmatrix}$

4. $\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}$

5. $\begin{pmatrix} 2 & -2 \\ 3 & 5 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 7 \\ 0 & -3 \\ 5 & 2 \end{pmatrix}$

6. $(4 \quad -3 \quad 2) + (3 \quad 5 \quad 5)$

7. Find $\begin{pmatrix} -2 & -1 \\ 1 & 5 \end{pmatrix} + \begin{pmatrix} -4 & -3 \\ 2 & 1 \end{pmatrix}$, then find $\begin{pmatrix} -4 & -3 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} -2 & -1 \\ 1 & 5 \end{pmatrix}$.

8. Given $A = \begin{pmatrix} 2 & -1 \\ 0 & 5 \end{pmatrix}$, find $3A$.

9. Given $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$, find $2A$ and $5A$.

10. Given $A = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 12 & -6 \\ 18 & 30 \end{pmatrix}$, express B in terms of A.

11. Given $A=\begin{pmatrix}1\\0\end{pmatrix}$, $B=\begin{pmatrix}5\\3\end{pmatrix}$, find $A+B$ and $2A-3B$

12. Given $A=\begin{pmatrix}-2 & 1\\3 & -5\end{pmatrix}$, $B=\begin{pmatrix}2 & 8\\-4 & 3\end{pmatrix}$, find $A+B$ and $B+A$

13. Given $A=\begin{pmatrix}-2 & 1 & 0\\3 & -5 & 2\end{pmatrix}$, $B=\begin{pmatrix}2 & -1 & 0\\-3 & 5 & -2\end{pmatrix}$, find $A+B$ and $2A-B$

14. Given $A=\begin{pmatrix}1 & -2 & 4\\2 & 5 & 6\\3 & 7 & -5\end{pmatrix}$, $B=\begin{pmatrix}-2 & 5 & 4\\8 & -6 & 6\\3 & 7 & -5\end{pmatrix}$, find $A+B$, $2A-B$ and $A-2B$

- Multiplication of two matrices in general, $A \bullet B = P$; order: $m \times e \bullet e \times t = m \times t$.

Example 6: Find the matrix AB and BA for the following matrices:

(a) $A=\begin{pmatrix}1 & 3\\-2 & 1\end{pmatrix}$ *and* $B=\begin{pmatrix}2 & -1\\1 & 0\end{pmatrix}$

(b) $A=\begin{pmatrix}2 & 0\\-1 & 3\\1 & -2\end{pmatrix}$ and $B=\begin{pmatrix}-1 & 1 & 3\\4 & -2 & 1\end{pmatrix}$

Exercise 2

1. $(1 \;\; 2)\begin{pmatrix}5\\6\end{pmatrix}$ $\quad (17)$

2. $\begin{pmatrix}3\\5\end{pmatrix}(-1 \;\; 3)$ $\quad \begin{pmatrix}-3 & 9\\-5 & 15\end{pmatrix}$

3. $\begin{pmatrix}-4 & 0 & -2\\5 & 9 & 6\end{pmatrix}\begin{pmatrix}3 & -8\\-1 & 4\\7 & 2\end{pmatrix}$ $\quad \begin{pmatrix}-26 & 28\\18 & 88\end{pmatrix}$

4. $\begin{pmatrix}3\\-2\end{pmatrix}(2 \;\; 4 \;\; -5)$ $\quad \begin{pmatrix}6 & 12 & -15\\-4 & -8 & 10\end{pmatrix}$

5. $\begin{pmatrix}2 & 4\\5 & 6\\-3 & 7\end{pmatrix}\begin{pmatrix}-1 & 3\\2 & -5\end{pmatrix}$ $\quad \begin{pmatrix}6 & -14\\7 & -15\\17 & -44\end{pmatrix}$

6. $(4 \;\; 2 \;\; 10)\begin{pmatrix}3 & 5\\1 & -4\\-2 & 6\end{pmatrix}$ $\quad (-6 \;\; 72)$

4. Determinant of Matrices

- For a 2×2 matrices $\begin{pmatrix}a & b\\c & d\end{pmatrix}$, the determinant $\begin{vmatrix}a & c\\c & d\end{vmatrix} = ad - bc$.
- For a 3×3 matrices $\begin{pmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{pmatrix}$, the determinant

$$\begin{vmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{vmatrix} = a_1\begin{vmatrix}b_2 & c_2\\b_3 & c_3\end{vmatrix} - b_1\begin{vmatrix}a_2 & c_2\\a_3 & c_3\end{vmatrix} + c_1\begin{vmatrix}a_2 & b_2\\a_3 & b_3\end{vmatrix}$$

$$= a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

Example 7: Find the determinant for the following matrices

(a) $\begin{pmatrix} -4 & 2 \\ 1 & 3 \end{pmatrix}$ *(b)* $\begin{pmatrix} -2 & 3 \\ -3 & -1 \end{pmatrix}$ *(c)* $\begin{pmatrix} 1 & -2 & 3 \\ 2 & 3 & -4 \\ 3 & -2 & 5 \end{pmatrix}$

Exercise 3: Find the value of determinant for the following matrices.

1. $\begin{pmatrix} 2 & -3 \\ 5 & 7 \end{pmatrix}$
2. $\begin{pmatrix} -6 & -7 \\ -8 & -9 \end{pmatrix}$
3. $\begin{pmatrix} 12 & -20 \\ -21 & 35 \end{pmatrix}$
4. $\begin{pmatrix} 1 & 5 & 1 \\ 1 & 6 & 3 \\ 9 & 8 & 9 \end{pmatrix}$
5. $\begin{pmatrix} 3 & 1 & -2 \\ 0 & -1 & 1 \\ 4 & 2 & 5 \end{pmatrix}$

5. Inverse Matrix

- **Identity matrix,** $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where the elements along the principal diagonal are all 1's while the other elements are all 0's.
- **Inverse matrix** for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. The expression $ad-bc$ is called the determinant of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and is denoted by det M or $|M|$ or $|ad-bc|$
- $AA^{-1} = I$ and $A^{-1}A = I$

Example 8: State whether inverse matrix exist in the following matrices. If yes, calculate the inverse matrix.

(a) $\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$ *(b)* $\begin{pmatrix} 4 & -6 \\ -6 & 9 \end{pmatrix}$

Example 9: If the inverse matrix doesn't exist, find the value of x *in* $\begin{pmatrix} 5-x & 4 \\ 2 & 3-x \end{pmatrix}$

Example 10: Given that $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$, *find*

(a) A^{-1} *(b) if* $AX = B$, *find* X *(c) if* $YA = B$, *find* Y

Exercise 4: Find the inverse matrix for each of the matrices given below:

1. $A = \begin{pmatrix} 3 & 2 \\ 4 & 2 \end{pmatrix}$ $\begin{pmatrix} -1 & 1 \\ 2 & \frac{-3}{2} \end{pmatrix}$
2. $B = \begin{pmatrix} 9 & 7 \\ 5 & 4 \end{pmatrix}$ $\begin{pmatrix} 4 & -7 \\ -5 & 9 \end{pmatrix}$
3. $C = \begin{pmatrix} 6 & 4 \\ -4 & -3 \end{pmatrix}$ $\begin{pmatrix} \frac{3}{2} & 2 \\ -2 & -3 \end{pmatrix}$
4. $D = \begin{pmatrix} 3 & 7 \\ -1 & -3 \end{pmatrix}$ $\begin{pmatrix} \frac{3}{2} & \frac{7}{2} \\ \frac{-1}{2} & \frac{-3}{2} \end{pmatrix}$

6. Solving Simultaneous Equation using Matrix

Example 11: By using matrix method, solve $\begin{cases} 4x+7y=3 \\ 2x-5y=-7 \end{cases}$.

Exercise 5:

1. Solve the following simultaneous equation using the matrix method.

a) $\begin{matrix} 4x+9y=31 \\ -2x+5y=13 \end{matrix}$ 1,3

b) $\begin{matrix} x+y=4 \\ 3x-y=14 \end{matrix}$ 2,2

c) $\begin{matrix} 3x-2y=19 \\ 4x+5y=-13 \end{matrix}$ 3,−5

d) $\begin{matrix} 4x=8-3y \\ 8x=21-9y \end{matrix}$ $\frac{3}{4},\frac{5}{3}$

e) $\begin{matrix} 3x+5y+4=0 \\ x+3y-6=0 \end{matrix}$ −10.5,5.5

f) $\begin{matrix} 2x+3y=3 \\ -4x-4y=2 \end{matrix}$ −4.5,4

2. Given $P\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find P. $\begin{pmatrix} 2 & -1 \\ \frac{-5}{2} & \frac{3}{2} \end{pmatrix}$

3. Given $Q\begin{pmatrix} 4 & -9 \\ -2 & 5 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find Q. $\begin{pmatrix} \frac{5}{2} & \frac{9}{2} \\ 1 & 2 \end{pmatrix}$

4. Find the inverse matrix of $A=\begin{pmatrix} 2 & 4 \\ -3 & -7 \end{pmatrix}$. Hence solve the system equations $\begin{cases} 2x+4y=-2 \\ -3x-7y=6 \end{cases}$ $\begin{pmatrix} \frac{7}{2} & 2 \\ \frac{-3}{2} & -1 \end{pmatrix}$ 5,−3

5. Find the inverse matrix of $P=\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$. Hence solve the system equations $\begin{cases} x-2y=-12 \\ 3x+4y=34 \end{cases}$ $\begin{pmatrix} 0.4 & 0.2 \\ -0.3 & 1 \end{pmatrix}$ 2,7